THE
MATHEMATICAL GAZETTE.

EDITED BY
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WITH THE CO-OPERATION OF
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The Mathematical Association.

THE Annual Meeting of the Mathematical Association was held at the London Day Training College, Southampton Row, London, W.C. 1, on Wednesday, 1st January, 1919, at 5.30 p.m., and Thursday, 2nd January, at 10.0 a.m. and 2.30 p.m.

WEDNESDAY, 5.30 *p.m.*

(1) **ADVANCED SECTION**: "The Graphical Treatment of Differential Equations," by Dr. S. Brodetsky.

THURSDAY, 10 *a.m.*

(2) The following Report of the Council for the year 1918 was distributed, taken as read, and adopted:

During the year 1918, 48 new members have been elected, and the number of members now on the Roll is 663. Of these 7 are honorary members, 38 are life members by composition, 37 are life members under the old rule, and 581 are ordinary members. The number of associates is about 150.

The Council regret to have to record the deaths of Professor O. Henrici, F.R.S., an Honorary Member of the Association and formerly Professor of Mechanics and Mathematics in the City and Guilds of London Central Institute; Dr. F. Hodson, Headmaster of Bablake School, Coventry; Baron D. Kikuchi, formerly Minister of Education and President of the Imperial University of Tokyo; Naval Instructor J. S. Robinson, R.N.; Mr. W. P. Workman, Headmaster of Kingswood School, Bath; and Mr. W. E. Mullins, for thirty-eight years a Master at Marlborough College.

The Council again recommend that the regulations which govern the elections of the Teaching Committees be temporarily suspended, and that the term of office of the existing Committees be extended for one year, or until such time as may seem convenient for an election to be held.

The Report of the Sub-Committee on the Teaching of Mechanics was issued to the members of the Association in the *Mathematical Gazette*, Oct.-Dec. 1918.

Professor T. P. Nunn, M.A., D.Sc., retires at this meeting from the office of President. The Council, in the name of the Association, desire to record their deep sense of the valuable services which Professor Nunn has rendered to the Association during his two years of office, and to thank him very cordially for his personal and unfailing attention to its interests. As Professor Nunn's successor the Council have the pleasure of nominating Professor E. T. Whittaker, M.A., Sc.D., F.R.S., Professor of Mathematics in the University of Edinburgh, to be President for the years 1919 and 1920. They also nominate Professor Nunn to be a Vice-President of the Association.

Mr. F. C. Boon and Captain W. M. Roberts retire from the Council by rotation, and are not eligible for re-election for the coming year. The members present at the Annual Meeting will be asked to nominate and elect others to fill the vacancies.

The Council again desire to acknowledge the indebtedness of the Association to Mr. Greenstreet for his services as Editor of the *Mathematical Gazette*, and to offer their thanks to the authorities of the London Day Training College for their kindness in affording accommodation for the Annual Meeting, and for the meetings of the Council and of the Committees which have been held during the year.

(3) The Treasurer's Report for the year 1918 was approved.

(4) The following Report was presented by the Teaching Committee:

The work of the Committee has again been hampered during the past year by the continuance of the war. Nevertheless some valuable work has been accomplished. The Sub-Committee appointed to draw up a report on the teaching of Mechanics has completed its labours, and the report has been issued to members of the Association as a number of the *Gazette*. In this connection the Committee wish to record their appreciation of the work of Mr. Goodwill, to whose energy and ability the report is mainly due. Another valuable piece of work which has been carried on during the year is the compilation of a list of books suitable for mathematical libraries for schools and teachers. A first list has been circulated with the *Gazette*, and suggestions for improvement have been invited from members. When these have been received a

final list will be issued. The Association is greatly indebted to Dr. Milne for the valuable work he has done in this connection.

A Vigilance Sub-Committee has been appointed chiefly for the purpose of scrutinising Examination questions and dealing with criticisms and suggestions relating thereto. The Secretary of this Sub-Committee is Mr. G. W. Palmer, Christ's Hospital.

(5) The Re-election of the Teaching Committees. It was proposed and *agreed* that the existing Committees remain in office for another year, or until such time as may seem convenient for an election.

(6) The Election of Officers and Council for the year 1919. Dr. S. Brodetsky and Mr. Eric H. Neville were elected members of the Council in the place of Mr. F. C. Boon and Capt. W. M. Roberts, who retired by rotation.

Professor E. T. Whittaker, M.A., Sc.D., F.R.S., Professor of Mathematics in the University of Edinburgh, was elected PRESIDENT for 1919 and 1920. The retiring President, Prof. T. P. Nunn, was elected a Vice-President of the Association.

(7) **GENERAL SECTION** : "The work of the Mathematical Association in assisting the application of Mathematics to Industry," by Dr. W. P. Milne.

(8) "The Teaching of Geometry to First-Year Pupils," by Mr. Basil A. Howard.

THURSDAY, 2.30 p.m.

(9) The President's Address *: "Astronomy as a School Subject." A discussion was opened by Mr. W. G. W. Mitchell, who was followed by Sir F. Dyson, the Astronomer Royal; Father Cortie of Stonyhurst; Mr. C. E. Livesey, of Bootham School, York; the Rev. W. W. Holdgate, Headmaster of Sutton Valence School, and others.

(10) "Cubic graphs of the form $y=ax^3+bx^2+cx+d$," by Mr. A. Lodge.

Members and Associates of the Mathematical Association were cordially invited to be present at the discussions at the meetings of the Association of Public School Science Masters, which were held at the London Day Training College on Tuesday, 31st December, 1918, and Wednesday, 1st January, 1919.

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* This was rather a *causerie* than an address; it was illustrated by an exhibition and explanation of a large number of wonderful home-made models.

MATHEMATICS AND THE PIVOTAL INDUSTRIES.

By William P. Milne, M.A., D.Sc.

The Mathematical Association existed first of all as a society for the improvement of geometrical teaching in the schools, and had as its avowed intention the infusion of a spirit of sweet-reasonableness into abstract geometry. Whether the Association aimed in the first instance at the unconditional abdication of Euclid is difficult to say, but at all events such was the final result. So successful were the labours of the Association in the realm of geometry that it resolved to extend its activities into the domains of Arithmetic and Algebra. To these were quite recently added the Calculus and Mechanics; but, broadly speaking, it must be said that its activities have been almost entirely confined within the somewhat narrow limits of the Secondary Schools. The contention of this communication is that great and clamant opportunities of service lie outside the walls of our more academic Secondary Schools, and it is proposed to deal with some of those newer spheres of activity.

The object of the present paper is two-fold—firstly, to discuss the importance of the Mathematical Association lending a helping hand towards drawing up suitable mathematical syllabuses for the various "pivotal" industries, such as Agriculture, Commerce, Engineering, Mining, etc.; and secondly, to see whether the Mathematical Association itself, and indirectly the Secondary Schools, would not benefit by occasionally viewing Mathematics from the different view-point of Industry rather than by focussing the whole attention on the academic aspect of the subject.

Consider first of all Agriculture. In the recent Parliamentary Campaign, the improvement, not only of agricultural conditions of tenure, but also of suitable methods of farming were openly advocated, so that as many people of healthy life as possible should be maintained on the land and the maximum of production per acre should be obtained. Improvement of agriculture is no new subject, and too many farmers have been prone, like their counterparts in all other professions, to take the line of least resistance and follow the well-beaten tracts consecrated by hoary antiquity. They have too often exhibited the spirit of the old Scotch gravedigger, who on his death-bed cautioned his son as follows: "Noo, Jock! I winna be lang here, sae I'll jist gie ye twa bits o' advice. Keep yer e'e on the meenister, howk (dig) deep an' resist a' impruvments." The theme of agricultural reform is also dealt with in Sir Walter Scott's *Pirate*, where it will be remembered that Triptolemus Yellowly and his sister Babie were transported from the ultra-civilisation of Kincardine or the Mearns to the Shetland Islands in order to proselytise the natives on the virtues of new-fangled coulters, spades, and harrows, thereby altering, as Norna of the Fitful Head pointed out, "the implements of our fathers from the ploughshare to the mousetrap." Truly a dangerous experiment and thankless task!

Another stage in the progress of agricultural reform has now been reached. The advantages of ploughs, harrows, potato-lifters, reapers and binders are nowadays taken for granted as axiomatic, and farmers are called upon to face a much higher order of subtleties, in the shape of Government requirements *re* percentage strength of milk, the disposition of various elements in the soil, the mixture of chemicals in the manufacture of manures and the proper adaptation of such manures to the soil of the individual farm. All this involves an adequate knowledge of Chemistry, Physics and Mathematics. The traditional figure of "Hodge," as he is depicted in the novels with his slouching gait, impervious countenance, dull wit and drawling voice, must perforce disappear from the English landscape and be replaced by a figure more akin to the "Economic Man" of the first chapter of every treatise on Political Economy

—a being active, alert, aggressive, with a mind well-stocked with scientific and mathematical principles, and ready to use them at the psychological moment. It is with such ideals in view that the country grammar schools and their modern counterparts, the rural secondary schools, are slowly but surely modernising themselves. They realise that 60 or 70 per cent. of their pupils will devote themselves to agricultural pursuits, and that therefore they should be discharged from the school, not entirely void of all those scientific notions which they are likely to be called upon to use in their life's work, in the furtherance of their agricultural calling. Not only is there the utilitarian aspect, but it is being more and more found that education ought to be based as far as possible on the phenomena with which the young child or boy is surrounded. By continual reference to the concrete, educational principles acquire precision and vigour. It is perfectly certain that the youthful farmer of 10 to 12 years of age will not be much enlightened by being told that a point is "that which has position but not magnitude." It is reported that in one of Lord Kelvin's classes, the famous natural philosopher asked of a Highland student, "Mr. MacTavish, what is a point?" The Celt replied after some thought and hesitation, "Sir, a point is a '*dab*.'" It is to be feared that in the minds of most boys—and grown men too, for that matter—a point will remain a *dab* to the end of the chapter. Again, what boy is not struck dumb with terror when the teacher begins his explanation of a "flat" surface, with the awe-inspiring words "*A plane superficies or surface.*" To the boy, the idea of planeness is derived from the concrete experience of blackboards, barn-doors, surfaces of ponds, etc., and no amount of abstraction or scientific verbiage will sharpen or vivify that acute concrete conception. Mathematics is thus destined to play a great and ever-increasing part in the farmer's life, and it seems desirable that the Mathematical Association should play its appropriate part in the great work of Agricultural Reconstruction.

Let us pass next to the department of Commerce. If one goes for a cycle run in the country and enters a small road-side house for a glass of milk, one finds on enquiring for the price that the reply will be one of three things:—(1) a definite sum of money; (2) Your pleasure, Sir; (3) You're welcome to it. Here we see Commerce in its most rudimentary form, baffling any mathematical law. If we compare with this the present international food situation, where all the populations of the earth have been numbered and all the stocks of corn, oil and wine counted, we see at once that we are living in a world where men's actions are being subjected as never before to the deductions of mathematical rigour. Nevertheless, the extreme elements of Mathematics have from time immemorial played an obvious and intimate part in Commerce. The famous treatise on *Directions for knowing all Dark Things*, by Ahmes the Egyptian priest, is really a commercial treatise on Fractions. Every booking-clerk in a railway station uses unconsciously the Austrian Method of Subtraction when he gives one change for 13/8 out of a pound by laying down 4d. and counting 14 shillings and then other 6 shillings to complete the pound. Every upholsterer possesses great facility in the method of Practice when he counts out the price of 15 chairs at £2 10s. 6d. each by estimating their price at £2 each, then at 10/- each, then at 6d. each and adding up to get the total. This sort of work requires a little experience and much repetition at the receipt of custom to be able to be performed expeditiously. A boy may be very awkward in dealing with such commercial processes when he first starts, but he soon gets up to it. A country grocer once condemned root and branch the whole system of British Education and Methods of Education, because a new apprentice that came to him did not know the difference between a florin and half a crown. The object of a liberal education is not to teach the minute technical details of all trades and professions, but to put the pupil in possession of such general principles that he will be able

when called upon, to learn intelligently the technicalities of whatever trade or vocation he adopts. Even quite elementary commercial transactions may, however, call for the exercise of distinct mathematical knowledge and ability. Consider for example the "Method of Nine Multiples"; suppose that a certain wholesale firm sells carpet sweepers in huge consignments at £2 13s. 7½d. each. If we decimalise this to (say) eight places, we can easily calculate the following and keep it for reference to all time:

Number of brushes.	Price.
1	£2·68229167
2	£5·36458333
3	£8·04687500
4	£10·72916667
5	£13·41145833
6	£16·09375000
7	£18·77604167
8	£21·45833333
9	£24·14062500

If now an order come to the business for (say) 869 brushes, these numbers are all on slips and are set below one another thus:

Price of 800 brushes = £2145·8333
Price of 60 brushes = £ 160·9375
Price of 9 brushes = £ 24·1406

Price of 869 brushes = £2330·911
i.e. £2330. 18s. 2½d.

Passing beyond the limits of the most rudimentary commercial transactions, one finds that the complexity of modern commerce calls for experts in various departments of Mathematics, requiring knowledge from the most elementary to the most advanced. Think of the statistical departments of some of the great Universities, preparing students for studying and interpreting social phenomena on a large scale—the effect of wages on birth-rate, the inter-relations of the export and import trade with such an apparently remote phenomenon as the marriage-rate, and so on. At least one University has a fully equipped Mathematical Department for the training of Actuaries. It is felt on all sides that the day of the man who can merely "turn up the tables" is past and gone. He must be able to know how the tables are constructed, and if possible to modify and improve the processes laid down in them. Many a modern banker is driven nearly crazy by his paucity of mathematical ideas in dealing with the various terms and rates of interest in the shoal of Victory Loans, Liberty Loans, War Bonds, etc., etc., that confront him nowadays as April 5th draws near for his clients' making up their annual income returns.

Is there not a golden opportunity here for the Mathematical Association to help in training the young men of this country who are to take up some branch of commerce as their vocation in life? Is not every high-souled mathematical teacher filled with an involuntary wave of melancholia when he is called upon by not a few of the Commercial Examining Bodies to teach "Recurring Decimals," and the methods of juggling therewith? Can the Mathematical Association do nothing, as the recognised leader of mathematical teaching thought in this country, to lay the ghost of that once familiar spirit of school arithmetics and—even yet—commercial syllabuses of knowledge? Again, has the Mathematical Association done anything to further the burning question of the adoption of the Decimal System? Have its members even clear ideas on the questions at issue? Has the Mathematical Association made any attempt to organise a joint-meeting, for example, with Commercial Magnates, so that one of those heart-to-heart talks may ensue, which are the order of the day?

These are merely typical concrete instances of great spheres of modern activity in which mathematics plays a great and ever-increasing part, and in the furtherance of which the Mathematical Association has as yet taken no appreciable share.

Apart altogether from the great contribution which the Mathematical Association could make to the adaptation of Mathematics to the requirements of modern industries, the older and more academic aspects of the subject as upheld by the Mathematical Association would be fertilised and reinvigorated in our secondary schools, by teachers being present at and taking part in discussions on the Mathematical Syllabuses of the various Industries. There is no doubt whatever that a much larger proportion of the population are learning to think in terms of Mathematical Symbols. One has only to turn up a copy of the *Times* or other modern newspaper to realise this. Tabulated increases or decreases are nowadays set forth with the signs + and − attached. I can hear Professor T. P. Nunn say to himself just now, "Ha ! the Anglo-Saxon race has at last tumbled to the notion of *Directed Numbers*." During the terrible submarine menace, losses and gains in shipping were regularly set down in the papers in terms of graphs, and I am not at all sure but that even members of Parliament were asked to contemplate that question in terms of squared paper. Why do we not teach the practical applications of "Correlation Graphs" in our schools; the newspapers use them ? All these things are significant, and teachers would do well to dwell upon them.

If some degree of study were devoted towards the teaching of Industrial Mathematics in the Mathematical Association meetings, it might help to shed some light on the burning question of what to teach boys during the time of their prolonged school age. In the past, when attendance at school after 14 years of age was not compulsory, it was almost impossible to get boys to stay on at the secondary schools after the age of 16 at the latest. Those large and fortunate schools which are able to teach boys free of charge and to finance them in their start in life found the utmost difficulty in persuading parents to prolong their sons' school life, even under such favourable conditions as I have described. The reason assigned by the parents is usually the well-known one of, they themselves having left school at 12 or 13 years of age, being put straight to work and then being able to "buy up" all those highly educated people whom they see around them and whom they know to have stayed at school and University till their age was well advanced in the twenties. This argument is unanswerable in terms of the units which these parents insist on using. If we consider the boys themselves, we must frankly face the fact that a large majority of them were thoroughly bored with what they were being taught at school and were itching to start the real business of life. Apropos of this attitude of both boy and parent towards the subjects taught in the secondary schools after 15 or 16 years of age, we may quote the dictum of a well-known educational wit. He remarked: "Cromwell was no true democrat. He never thought of what the people liked, but always what was good for them." Have not teachers done the same ? This does not mean that schoolboys should always be pampered in their mental provender and never get anything except what appealed to their intellectual palates, but it does mean that some attempt should be made to teach them subjects which, while educating their faculties, at the same time will interest their senses and make them feel that they are learning something that will directly or indirectly be of use and advantage to them later in life. The schoolboy, in plodding along the tracks of his school career, tends only to look at his feet and the immediate utilitarian usefulness for his first job of what he is learning, while the teacher tends to keep his gaze fixed on the mountains afar in selecting the material he is to teach. Is there no mean between these extremes ?

If we read Mr Bertrand Russell's *Problems of Philosophy* we find that a common or garden kitchen table can be viewed from several standpoints. To the washerwoman it is a handy means of ironing clothes; to the furniture-dealer it is a commodity that will fetch a certain price; while to Mr. Bertrand Russell himself it is a psychological aggregate of sense-data impressions. So with Mathematics. Needless to say, the century-old attempt to present Mathematics from the last-named standpoint accounts to a large extent for the century-old boredom with which too many boys contemplate the subject.

Masters on the Army sides of schools frequently declare that they see fewer exhibitions of boredom, and they attribute this phenomenon to the fact that boys are working towards a definite object, and also that the subjects are presented in a less abstract form. Whatever be the true explanation, it is absolutely certain that something is wrong, so long as boys feel as they advance in their school course that the subjects they learn are utterly remote from the objects they have in view. Perhaps more frequent and intensive discussions at our meetings of the Mathematics of the various industries will fertilise our minds and help us towards some sort of satisfactory solution of this great educational problem, which is now thrust upon us on a gigantic scale, whether we will or no.

A proposal has been made, and is receiving widespread support, that the Mathematical Association should hold its winter meeting in London and a summer meeting in the provinces. In that way it is hoped to carry the principles and progress for which the Mathematical Association has always stood more intimately into the working life of the mathematical teachers in the various parts of England. Another end that a peripatetic meeting would serve is that the Association could meet (say) in a Mining District one year and study Mining Mathematics on the spot; in an Agricultural District another year and study Rural Mathematics in the fields; in a Seaside Town a third year and study Hydrostatics by the sad sea waves. One might thereby be able to realise for the first time that the mathematical subject of Hydrostatics has some connexion with water, and not only with vanishing triangles which prove that the "pressure at a point in a perfect liquid is the same in all directions." In this way teachers of mathematics for Engineers, for Seafaring men, for Commercial men, and so on might have the opportunity of meeting and discussing the special difficulties in each technical trade of teaching mathematics from the trade standpoint, and might even derive some assistance from those teachers whose standpoint is first and foremost academic. It must not be forgotten that a teacher of "Occupational Mathematics" has a two-fold calling which ordinary school teachers of the more formal mathematics have not got to face. The mathematical master in a secondary school has to teach his boys approximately what he himself has learned at the University, whereas the teacher of Industrial Mathematics has to know two subjects, namely, the technicalities of the particular occupation and the subject of mathematics itself.

In conclusion, I hope I am not giving away any Cabinet secrets, if I say that the Mathematical Association has definitely taken in hand the subject of the Mathematical Syllabuses of the various trades and occupations and will publish its findings in due course. Each subject, Agriculture, Commerce, Engineering, etc., will be dealt with by a Committee consisting of two permanent members of the Mathematical Association, together with three co-opted members from the particular trade. Let it be hoped that a great wave of public opinion and enthusiasm will stimulate and encourage this new movement of the Mathematical Association. W. P. Milne.

THE TEACHING OF GEOMETRY TO FIRST-YEAR PUPILS.

A Paper read to the Mathematical Association on 2nd January, 1919.

BY BASIL A. HOWARD, B.A.

No general agreement about a first-year syllabus in Geometry is possible unless there is some measure of agreement as to the aim we ought to have in teaching geometry at all. For to different aims there will correspond differences, not only in the manner of teaching the syllabus, but also in the actual matter taught. What, then, ought to be our aim ?

We may aim at training a boy to think clearly, and argue logically; to analyse any and every problem confronting him until he sees the real thing upon which that problem depends; to disentangle the essential from the incidental. In other words, we may regard geometry as a field for mental training. Other subjects should aim at this too; but in teaching geometry we have a specially rich opportunity for doing so.

Or we may take a different point of view. We may believe that geometry should be taught for its "usefulness"; that the mathematical master is merely an assistant to the Science and Engineering ones; and that his proper function is to supply that minimum of geometrical knowledge which is necessary for technical students. In other words, we may have a frankly utilitarian aim. And so we may be disposed to teach only those portions of geometry which seem likely to be capable of immediate practical application. Those who hold this narrow utilitarian point of view will find that it will lead them to extremes to which they may hardly be prepared to go. If the sole aim of geometry is to be "useful," why teach it at all to two-thirds of our pupils ? They will become journalists, barristers, doctors; or else grocers, clerks, railwaymen; on strictly utilitarian grounds none of these classes has much need of geometry. In which case, if we were honest with ourselves, we should banish it altogether from the Classical Sides of our Schools.—Incidentally, I suppose, we should abolish the Classical Sides themselves.

In so far as this view is held at all, it is probably the outcome of a reaction against an abuse of the opposite view; against the tedious formalism of the Euclideans, and the un-relation of geometry to real life that was implied in their teaching. It is altogether excellent that we should protest against these things; only, do not let us forget that it is possible to condemn these abuses and yet to hold that the true aim in teaching geometry is to afford a training in habits of accurate thought.

This is the standpoint of the present paper; that our chief aim should be the development of the reasoning faculty; insisting always that this can only be done by sweeping away the abuses of the Euclidean tradition, and giving a more human and less philosophic orientation to our teaching.

There are two well-defined systems of teaching geometry to-day. There is firstly the system which for shortness we shall call Euclidean, but which is here meant to include geometry as taught by those who introduce only minor changes in Euclid's sequence; and there is, secondly, the modern one, with its broad basis of axioms, a schedule which we shall shortly discuss.

It is often supposed that it is the Euclidean system which trains the reasoning faculty, the modern one which is content to provide a mass of so-called "useful" facts; and it is this glaring perversion of the true state of things which operates in the minds of cautious teachers when they condemn the new method.

Now, even supposing the Euclidean system to be completely logical, it seldom produced any real comprehension of logical reasoning in the mind of

the average boy. The better boys triumphed over the difficulties, and no doubt did eventually discover what geometry was really about; but they did it in spite of the method of teaching, not because of it. We read, for instance, that Hobbes, while waiting in a gentleman's library, "saw a book of Euclid lying open on the table, and read the enunciation of Euc. I. 47. 'By G—,' says he, 'but this cannot be!' So he read the demonstration, which referred him back to another, which he also read, and so on, till he was at last demonstratively convinced of its truth. This made him in love with geometry." Its effect on the mind of the average boy, however, was very different from this; it is much more nearly indicated by the story of the boy who, throughout his school course, laboured under the slight misapprehension that his master said, not "*ABC* coincides with *DEF*," but "*ABC* goes inside *DEF*." The majority of pupils, in fact, got in a state of hopeless confusion which could only end by prejudicing them against the study of geometry in every form. So that, even assuming the Euclidean system to be a logical one, it still failed in its object of training boys to reason logically.

But in point of fact, of course, the older system was very far from being a completely logical one. It started from an axiomatic basis that is not the ultimate basis at all; and the only reason why Hobbes was convinced of its truth was that it did not occur to him to question those axioms on which Euclid based the subject. But if he had continued his backward process a little further, he would have found that Euclid's axioms could be reduced to dependence on others more ultimate still. This is not the place to go into the question of what the ultimate basis of geometry really is; it is enough to say that it is metaphysical, and not a possible basis from which to start boys at all. There is, therefore, nothing sacrosanct about the Euclidean basis at all, and nothing illogical about adopting a different one.

The modern system of geometry, then, with its broader basis of assumed truth, is no less logical than Euclid; and it has the overwhelming advantage that it speaks a language understanded of the schoolboy. For if we are to be true to our aim of training boys to think logically, it is no use giving them a basis they cannot appreciate, or confronting them with a series of propositions the need for which they do not feel. We shall never get them to think at all, much less to think logically, unless we give them work they can understand; unless, indeed, we are able to arouse their interest. From whatever angle we approach this subject, we are led to this as one of our fundamental principles. We shall not, of course, seek to enlist their interest by denuding geometry of every vestige of difficulty, and presenting them with work they can do without any exertion at all; such an attitude is quite as likely to kill interest as to quicken it. But equally, we must strive to give them from the very start a sense that the subject is a real and living one, and that all sorts of interesting and important things can be discovered by this new game of "proving" things.

There is, fortunately, some measure of agreement as to what the basis of our deductive work should be. But there is some preliminary work of a practical nature to be done first, before we can proceed to deduction. It will be convenient now to outline a schedule, and then discuss it in the light of the foregoing.

A First-Year Schedule in Geometry.

A. **The relations between solids, surfaces, lines and points.**
Illustrated from models of the regular solids.
Plane and curved surfaces.
Plane figures.

Angles: measurement: use of protractor: cardinal points of mariner's compass.
Supplementary: complementary: vertically opposite angles.
The "parallel axiom."
"Chart" problems: leading to
The practical construction of triangles from sets of data.
The three tests for congruence of triangles.

B. Angle sum of a triangle:
Theoretically: also practically.
Calculation of the interior angles of the regular polygons.

Deductive work starting from the above basis:
Riders on congruence: properties of isosceles and equilateral triangles.
Constructions for bisecting lines and angles, and drawing perpendiculars and parallels: with proofs of these constructions.
Elementary properties of the parallelogram, rectangle and square.
One or two circle properties.

Area of rectangle and triangle:
Theoretically: also numerical work: triangulation: Surveyor's Field Book.
Triangles and parallelograms which are equal in area.
Pythagoras' Theorem.

The portion "A" of this schedule outlines the preliminary work which must precede deduction. The aim of teaching it will be to get pupils clearly to understand the machinery with which they will have to work; and the teacher will use whatever methods he thinks most suited to the capacity of boys of twelve. But in any case he will not confuse them by offering, at the outset, precise definitions of all the terms he is using. If such definitions are needed at all, they should come after, and not before, his pupils have had some practical experience of the subject. "A point has position, but no magnitude"—what purpose is served by insisting that boys shall learn that? If a point has really no size, no boy will ever believe that it exists at all; that is the end of the matter for him. Or again: "A straight line has the same direction from point to point." Every boy knows what a straight line is; and, in any case, such a definition amounts to no more than a description of one vague word in terms of another vague word. It is much better to tell him that a straight line is the shortest distance between two points, and have done with it. One imagines that nowadays, even in a public examination, a "proof" of this fact will hardly be demanded. In short, the relations of points, lines, surfaces and solids to one another will be much more clearly grasped by an actual examination of cubes, cones and cylinders than by the learning of any number of so-called definitions.

Angles will be treated by the rotation method, and no restriction will be placed on their size. A right angle will simply be one quarter of a complete turn. It follows naturally that if one straight line falls upon another straight line, the sum of the angles so formed is two right angles; any attempt to make a formal proof of so simple a thing is likely to discredit the whole idea of a "proof" in the mind of a boy of twelve. He will be impressed by the value and importance of a formal proof in proportion as he sees that such proofs do add materially to his knowledge. This is not to say that, at a somewhat later stage of his work, he is to be excused from proving any fact which his intuition tells him is probably true; he will later come to realise that intuition alone, valuable as a servant, is apt to be very dangerous if allowed to become master. But that stage is not yet; and for the present we shall aim at reducing the proofs of more or less obvious properties to the lowest possible number. For the same reason, it seems undesirable to elevate the proof of the equality of vertically opposite angles, which we shall take next, into a Theorem.

We can now proceed to the fact that parallel lines make equal angles with any line which crosses them—our 'parallel axiom.' The assumption of this has enormous advantages over the older method of proving it; at one sweep it does away with the series of tiresome—and unsatisfactory—propositions by which Euclid had to lead up to it. And it is, to a boy, a perfectly natural and reasonable assumption to make, accompanied as it will be by the familiar argument about a man walking along a road, turning to the right through a certain angle into a cross-road, and turning back again through the same angle into a parallel one. We may be very sure that boys of twelve will not seek for any logical flaw in such a demonstration!

All this time the boy will, of course, be occupied with much practical work, in the course of which he will be acquiring two things; familiarity with these conceptions, and accuracy in the use of his instruments. He will now proceed to the working of "chart" problems, and will realise, at first almost unconsciously, how to construct triangles from various sets of data. The tests for congruence will arise naturally out of this work; when he has mastered these his axiomatic basis will be complete. These tests can be dealt with in two or three ways. The way suggested in the Board of Education Circular 851 is to draw a triangle on the blackboard and ask what elements must be given in order to make a copy of it; but there is nothing really difficult for a boy in the older method of superposition. What made it so repellent formerly was the long and tiresome "talk" in the proofs, coupled with the use of unfamiliar words; the physical labour involved in writing out the proofs distracted attention from the real point of them. On the blackboard they can quickly and easily be explained without the risk of obscuring the essentials by a mass of words.

We have now enough material to form the basis of our logical structure, and can embark on the second section of our schedule. Our first theorem will be the angle sum of a triangle. It will be a good theorem to begin with, because it expresses what is, to a boy, quite a remarkable fact.—(As a minor point, this theorem is really needed to complete the third congruence test; but it seems advisable to take congruence first, in order to have all our axioms complete before starting the formal work.)—We can now easily calculate the angles of the regular polygons, and hence construct them; and these polygons, by the way, furnish quite a good field for simple riders on congruence.

From now onwards the work follows a straightforward course; it can be made quite logical without, with care on the part of the teacher, losing its interest for boys. The property of the isosceles triangle, the constructions for bisecting lines and angles, the parallelogram properties, are all simple deductions from the congruence tests, and should be treated as such. Some reformers would have us assume all these things; and if their proofs offered any real obstacle to a young boy, there would indeed be a strong case for assuming the properties, or at any rate delaying their proofs. In point of fact, however, they are easy applications of congruence, and the majority of boys take considerable interest in proving them. Besides, a good deal of practice in applying the congruence tests is necessary; and it is surely better to give boys that practice by setting them to prove things which really do matter, than by giving them hosts of examples which lead nowhere.

Two conditions, however, are necessary if the remainder of the schedule is to be made really interesting to boys.

These conditions are: first, that the teacher makes every contact possible with every-day experience, illustrating the work at every turn by all the practical applications he can think of. This will be so generally agreed that it is unnecessary to labour it.

Secondly, that we shall not seek to introduce much detail, but shall content ourselves with filling in the broad outlines of the structure only. In teaching History to boys of twelve, we should not tell them every fact we know about one period before proceeding to the next; and the same idea will guide us in geometry. For instance, we might with complete logical soundness prove in a first-year course on these lines that the medians of a triangle are concurrent; but there would be nothing unsound in postponing it, which is, of course, what we shall do. Again, when dealing with areas we shall not attempt to cover more than one or two propositions of Euclid's Second Book, and these will be co-ordinated with algebra in the frankest possible way, the old verbose notation being drastically simplified. This principle will be conceded up to a point by every teacher; it is merely a question of just how far we shall go; and my present contention is that we can and ought to go much further in the

matter of cutting down the number of propositions in a first course than any of the usual text-books indicate.*

Neither of these conditions need in any way impair the logical development of the subject, nor serve as an excuse for inaccuracy on the part of either teacher or pupil.

The schedule outlined is quite as much as a boy of average ability is likely to cover in a first-year. Many of us would like to be able to introduce the important and attractive idea of a locus into a first-year course; we shall certainly aim at taking it early in the second year. And there are doubtless others who will wish to treat of similarity as soon as possible. It is not so very long ago, however, that similarity was reckoned as third or even fourth-year work; and we shall be doing very well if we manage to introduce it, side by side with trigonometry, towards the end of the second year. The omissions in the schedule, as compared with Euclid, are obvious: they consist of a whole series of minor propositions, some of which we shall have to come back to later, but the majority of which one hopes may be decently buried in this twentieth century.

The scheme may fairly be described as a medium one, and as such will share the fate of all medium schemes; it will be assailed both from the left and the right. It is criticism from the left which is more likely to be offered by members of the Mathematical Association; and in anticipation of that it may be pointed out that the scheme has one advantage which a more revolutionary one could not have: it offers a basis upon which teachers of varying opinions may be induced to unite. In many schools to-day it must happen that advocates of the new teaching are taking parallel sets with teachers of a more conservative type. If each teacher went his own way, the result at the end of the year, when promotions were due, would be chaos; there must undoubtedly, therefore, be a common syllabus; and the one suggested in this paper is one which, while embodying all the essential principles of the modern method, will do as little as is possible to outrage the feelings of believers in the older one.

To sum up, then, the argument of this paper. We are aiming primarily at developing the capacity for accurate reasoning; at getting boys to think things out for themselves; and only secondarily at any other aim. And we have far more chance of realising our aim by working on modern lines, and adopting the spirit and sequence of some such schedule as here outlined, than we have by working through a text-book of Euclid, or any of his modern paraphrases.

I think it will be relevant to add one final remark, arising out of the challenge of the present world situation. We do not need to be expert psychologists to realise that thousands of people to-day are little more than mental parasites. They take their opinions ready-made from their friends, or from newspapers; and so they are in bondage to any foolish catchword of the moment. We have made a great success of teaching people to read and write; we have yet to solve the problem of teaching them to think. But the civilisation of to-morrow is modelled in the schools of to-day; and so the task of the educationalist will be, not to tell his pupils what to think, but to teach them how to think; to make them critical of an argument, impatient of a non-sequitur. And, as mathematical masters, we shall not be making a very worthy contribution to the task if we limit our duty to that of increasing the level of technical ability among our pupils. It is up to us to attempt the more fundamental task of training the intelligence; in the belief that, if a boy learns to become accurate and logical in his work in the mathematical class, he will slowly and unconsciously take over these qualities into the region of every-day life.

BASIL A. HOWARD.

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* It is obvious that the problem confronting the teacher differs in many respects from that confronting the writer of the mathematical text-book.

CUBIC GRAPHS OF THE FORM $y=ax^3+bx^2+cx+d$

By A. Lodge, M.A.

The object of the paper is to call attention to some simple means of interpolating points in a cubic graph, and of indicating gradients without actual calculation.

For integer values of x, calculation of y is probably better than any geometrical method. The best plan for such calculation is, I think, as follows for each such value of x:

(1) Put down a,
(2) Calculate $ax+b=m$ (say),
(3) " $am+t$, *i.e.* ax^2+bx+c, $=n$,
(4) " $y=an+d$,

arranging the calculation thus:

$$x\left|\frac{a,\ b,\ c,\ d}{a,\ m,\ n,\ y}\right..$$

For example, if $y=x^3+5x^2+x-14$, and values are wanted between $x=-5$ and $x=+3$ inclusive:

	1+5+1−14		*Check by differences.*		
−5	1±0+1	−19			
			17		
−4	1+1−3	−2		−14	
			3		6
−3	1+2−5	+1		−8	
			−5		6
−2	1+3−5	−4		−2	
			−7		6
−1	1+4−3	−11		+4	
			−3		6
±0	1+5+1	−14		+10	
			+7		6
+1	1+6+7	−7		+16	
			+23		6
+2	1+7+15	+16		+22	
			45		
+3	1+8+25	+61			
values of x		values of y			

The gradients $y+3x^2+10x+1$ can be calculated similarly or by the following scheme:

x	−5	−4	−3	−2	−1	0	1	2	3
$3x+10$	−5	−2	+1	+4	+7	+10	+13	+16	+19
Gradient	26	9	−2	−7	−6	+1	+14	+33	+58

But though these values may be utilised for the graph, there are much simpler geometrical ways of obtaining some of the gradients, and there are interesting geometrical devices to be considered with regard to the rest, and for interpolating other points on the graph.

The first geometrical consideration to be utilised is that every straight line meets the curve at points, the sum of whose abscissae $=-\frac{b}{a}$; in this case $=-5$.

Let each line be designated by the abscissae of the points on the curve through which it passes; thus the following sets of points are collinear:

$-5-3+3$
$-5-2+2$
$-5-1+1$
$-5 \quad \pm 0$ (tangent at $x=0$)
$-4-1+0$
$-4-2+1$
$-4-3+2$
$-4-4+3$ (tangent at $x=-4$)
$-3-3+1$ (tangent at $x=-3$)
$-2-2-1$ (tangent at $x=-2$)
$-1-1-3$ (tangent at $x=-1$),

so that 5 gradients are obtained at once without calculation.

If points corresponding to $x=\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$, etc., were plotted, the gradients at some of them could similarly be found, for

$$\left.\begin{matrix} -\frac{1}{2}, & -\frac{1}{2}, & -4 \\ -\frac{3}{2}, & -\frac{3}{2}, & -2 \\ -\frac{5}{2}, & -\frac{5}{2}, & 0 \\ -\frac{7}{2}, & -\frac{7}{2}, & 2 \end{matrix}\right\} \text{are all sets of collinear points, with a pair of points coincident.}$$

Tangents for points outside the above ranges will be considered later.

With regard to interpolations, one method is to calculate one such halfway point, and then put in the others by the collinear method; but the following method is worth consideration:

The points where any line of the form

$$y=t^2(ax+b)+(cx+d)$$

cut the ordinates

$$x=\pm t$$

are on the curve, and by varying t, any number of pairs of such points can be speedily found.

All these lines radiate from the point S on the curve where the ordinate $x=-\frac{b}{a}$ cuts the line $y=cx+d$, which is itself one of the same radiant system, and is the tangent at the point T where the curve crosses the y-axis, *i.e.* where $t=\pm 0$.

These lines cross the y-axis at distances bt^2 above or below this tangent, according as b is positive or negative, and they cross any given ordinate at heights above (or below) this tangent given by at^2, multiplied by the distance of the ordinate in question from the radiant point S.

Thus, on the 4th ordinate from S, the distances above or below $y=cx+d$ are:

0	a	$4a$	$9a$	$16a$	$25a$	$36a$	$49a$	$64a$	$81a$	$100a$
for $\pm t=0$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5

since $x=-\frac{b}{a}+4$, and the required distance is $t^2(ax+b)=4at^2$.

Some of these intervals may be off the paper, but we can always fall back on the general collinear method when desired.

Another method is to draw the line $y=cx+d$, and *one* of the other lines

$$y=t^2(ax+b)+(cx+d).$$

Let A be the point where $x=t$ meets this second line, and let the ordinate of A meet ST in E.

Let P be any point on SA, and let $PL=t^2(ax+b)$, *i.e.* the part of the ordinate above ST, the abscissa of P being x.

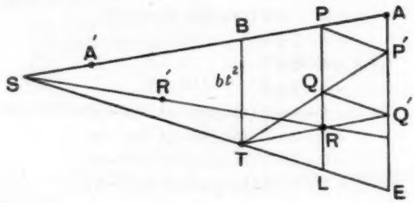

FIG. 1.

Draw $PP' \parallel$ to ST to meet AE in P', and join TP', cutting PL in Q, then

$$QL=\frac{x}{t}PL=xt(ax+b).$$

Now draw $QQ' \parallel$ to ST to meet AE in Q', and join TQ', cutting PL in R.

Then $$RL=\frac{x}{t}QL=x^2(ax+b).$$

$\therefore$ R is a point on the curve, and if SR is joined, a second point R' can be plotted as far to the left of the y-axis (BT) as R is to the right. Similarly, of course, A' exists corresponding in the same way to A.

One very important point, which can be easily found in this way, is the point of inflexion I, which is situated on the ordinate $x=-\frac{b}{3a}$, for the three coincident points $-\frac{b}{3a}$, $-\frac{b}{3a}$, $-\frac{b}{3a}$ are collinear, since their abscissae add up to $-\frac{b}{a}$. But there are other ways of locating I, as we shall see. Perhaps the best way is to *calculate* its ordinate and have done with it.

N.B.—In all this it is not necessary to restrict the line BT to the y-axis. Any ordinate will serve equally well, provided ST is so chosen as to be the tangent at T where the curve crosses it, and t is measured from T.

In particular, ST may be chosen to be the tangent at I, in which case both S and T coincide at I, and any radiant line, such as IA, cuts the ordinates at distances above or below this tangent $=at^2\times$ the distance of the ordinate from I, where t is the distance of A from the ordinate through I, and A itself is therefore at ordinate-distance at^3 from this tangent. By taking a couple of such points, the tangent at I can be simply constructed, and therefore I found.

Construction for the tangent at any point A.

Referring to the previous construction for a point R by a zigzag $PP'QQ'R$, if P were close to A, the lines TP', TQ' would be nearly parallel to TA, since P', Q, Q' would all be close up to A.

Hence, if we modify this construction by treating T as infinitely distant, and drawing $P'Q$ and $Q'R$ parallel to AT, starting from any point P on SA, R will be a point on the tangent at A.

Since, evidently, $QR=PQ$ we need not complete the zigzag, but after finding Q, double PQ, so obtaining R. Moreover, if TA is joined, cutting PL in K, Q can be best located by measuring AP' with the dividers, and adding this distance to PK.

In particular, if P be taken at B on the y-axis, K will be at T, and all we need is to draw $BB' \parallel$ to ST, cutting the A ordinate at B', measure AB' on the dividers, add it to BT, *i.e.* make $TQ=AB'$ on the y-axis, then double BQ to find R, the point where the tangent at A cuts the y-axis.

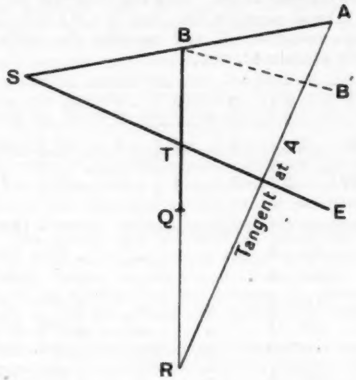

FIG. 2.

It is simple now to verify the expression for the gradient of this tangent. For if the A ordinate cuts ST in E,

$$AE=t^2(at+b) \quad \text{and} \quad BT=bt^2.$$

$$\therefore\ AB'=at^3.$$

$$\therefore\ 2BQ=2(at^3+bt^2).$$

Also the full ordinates of A, B are

$$at^3+bt^2+ct+d \quad \text{and} \quad bt^2+d.$$

$\therefore$ the distance of B below $A=at^3+ct$.

$\therefore$ the distance of R below $A=3at^3+2bt^2+ct$, and t is the abscissa-distance between A and R.

$\therefore$ the gradient$=3at^2+2bt+c$. Q.E.D.

There is, however, a far simpler method of constructing the tangent at any point A if we make ST coincide with the tangent at I. For in this case the sum of the roots obtained on any transversal is zero, *i.e.* $b=0$.

$\therefore$ B coincides with T, $AB'=AE=at^3$ and $TR=2AE=2at^3$.

$\therefore$ the tangent at A passes through the trisection point on IE furthest from I.

The great advantage of this last method is that it is easy to remember, whereas the other methods, though interesting, are liable to escape the memory.

This tangent method and the original three-point collinearity are both easy to remember, and the radiant method of interpolation is not difficult. The zigzag method is also fairly simple, and for close interpolation may be very useful. All chords through I are bisected, as is easily seen by the fact that if the I ordinate is taken as the y-axis, the sum of collinear abscissae$=0$.

This brings us to my last consideration, viz. the change of the equation of the curve as different ordinates are taken as y-axis, there being assumed no change in the x-axis.

We have seen that the sum of collinear roots $= -\frac{b}{a}$, and it is clear that each unit shift of the y-axis to the left adds 1 to each of the roots, and therefore adds 3 to their sum.

Also the gradient of the curve where it crosses the y-axis $=c$, since the line $y=cx+d$ is the tangent there. And the intercept on this axis $=d$.

If therefore we take as example the curve $y=x^3+5x^2+x-14$, where $a=1$, and the sum of the roots is -5, and consider the table of gradients and ordinates previously calculated, viz.:

x	−5	−4	−3	−2	−1	0	1	2	3
Gradient	26	9	−2	−7	−6	+1	+14	+33	+58
y	−19	−2	+1	−4	−11	−14	−7	+16	+61

Equation when $x=-5$ is taken as $x=0$, $y=x^3-10x^2+26x-19$,
" $x=-4$ " " " x^3-7x^2+9x-2,
" $x=-3$ " " " x^3-4x^2-2x+1,
" $x=-2$ " " " x^3-x^2-7x-4,
" $x=-1$ " " " $x^3+2x^2-6x-11$,
" $x=0$ " " " x^3+5x^2+x-14,
" $x=1$ " " " $x^3+8x^2+14x-7$,
" $x=2$ " " " $x^3+11x^2+33x+16$,
" $x=3$ " " " $x^3+14x^2+58x+61$.

The way in which the signs of the coefficients change as the y-axis moves to the right is very instructive.

GLEANINGS FAR AND NEAR.

25. *D.* All the time we lived at Chelsea we had constant intercourse with Lady Noel Byron and Ada, who lived at Esher. Ada was much attached to me, and often came to stay with me. It was by my advice that she studied mathematics. . . . Among my papers I lately found many of her notes, asking mathematical questions. Ada Byron married Lord King, afterwards created Earl of Lovelace, a college companion and friend of my son.—M.S.

[Lady Lovelace was the daughter of Lord Byron.]

It was during these meetings [at Turin] that my highly valued friend, M. Menabrea, collected the materials for that lucid and admirable description [of my Analytical Machine] which he subsequently published in the *Bibl. Univ. de Genève*, t. xli. Oct. 1842.

The elementary principles on which the Analytical Engine rests were thus in the first instance brought before the public by General Menabrea.

Some time after the appearance of his memoir on the subject in the *Bibliothèque Universelle de Genève*, the late Countess of Lovelace informed me that she had translated the memoir of Menabrea. I asked why she had not herself written an original paper on a subject with which she was so intimately acquainted ? To this Lady Lovelace replied that the thought had not occurred to her. I then suggested that she should add some notes to Menabrea's memoir ; an idea which was immediately adopted.—Babbage, p. 135.

REVIEWS.

An Elementary Treatise on Curve Tracing. By PERCIVAL FROST. Revised by R. J. T. BELL. Fourth edition. Pp. xvi+210. 12s. 6d. net. 1918. (Macmillan and Co.)

Doubtless many will welcome this new edition of an old friend of 1872, to which has been added an index and classified list of curves.

But perhaps the reviewer may be forgiven for a wish that Dr. Bell had undertaken the bolder task of entirely reconstructing Dr. Frost's material and bringing the whole up to date. It seems that the book was originally intended to provide interest and exercise for the student before he enters on the study of the calculus, mechanics, and the field of the Physical Sciences. Such a suggestion now appears to emanate from another world, and the dodges employed by Dr. Frost to avoid the use of the calculus are fortunately quite obsolete.

In reality the processes necessary for tracing the algebraic curve $f(x, y)=0$ are few in number: Find where the curve cuts $x=0$ and $y=0$: notice symmetry: solve, if possible, for x and y: equate to zero the terms of highest and lowest degrees to get the tangents at the origin and the directions of the infinite points: use Newton's diagram: find the asymptotes, their finite intersections with the curve, and the side on which the curve approaches them: express any point of the curve in terms of a convenient parameter. Few curves will defy tracing by means of these steps. If closer approximations are required at infinity, we can project the curve by means of the substitution x/y for x, $1/y$ for y, and use well-known methods for approximating to a curve at a finite point.

All this could have been condensed into a chapter or two, and Dr. Frost's diagrams would have served as illustrations of a fine set of examples. Space might then have been found for a proper discussion of singularities, etc.

The reviser's hope that "this edition will be found to be comparatively free from inaccuracies" unfortunately proves fallacious on a casual glance at the diagrams. Some errors, *e.g.* plate V. fig. 24, and VI. 17, are doubtless due to the printer. But III. 27; IV. 25, 26, 29 represent curves cut by a line in points whose number is greater than the curve's degree; while IV. 28 is also rather rough and ready. There is a tendency to draw cusps as though the tangents thereat were distinct, to exaggerate the rate of approach of a curve to its asymptote, and to misplace inflexions and their tangents. This is the more to be regretted, in that these are just the mistakes against which a beginner most needs warning.

The reviewer's copy tends to fall to pieces on being opened, which is doubtless due to war-conditions in the binding trade. But the printing and the facilities for reference are very clear and excellent throughout.

HAROLD HILTON.

Infinitesimal Calculus. By F. S. CAREY. Section I. 6s. net. Section II. 10s. net. 1917-1918. (Longmans, Green.)

The above text-book consists of some 360 pages and is issued in two volumes. Chapters I. and II. deal with the fundamental notion of Number, Function, Graph, Limit, and Continuity. Chapters III., IV., V., VI., VII. treat of such subjects as Differential Coefficient and its elementary processes and applications relative to the simpler functions, *i.e.* powers of x, polynomials, trigonometrical functions, etc. In Chapter VIII. the second differential coefficient is introduced and its connection with radii of curvature, acceleration, etc., explained. Chapters IX., X., XI. discuss Inverse Differentiation (leading at once to the necessity for introducing the Logarithm in dealing with $D^{-1}x^{-1}$) and the practical subjects of Areas, Volumes and Moments of Inertia. Chapter XII. disposes of Exponential and Hyperbolic Functions, Inverse Circular and Hyperbolic Functions. In Chapter XIII. the motion of a particle along an axis is discussed. In Chapter XIV. we find general theorems such as the Mean Value Theorem, as well as the Definite Integral. Polar Coordinates and

further geometrical properties are treated of in Chapter XV., and Partial Differentiation, together with Double Integration, in Chapter XVI. Expansions in Power Series with applications to Curve-Tracing appear in Chapter XVII., and more geometry—Envelopes, Evolutes, Roulettes—in Chapter XVIII. Chapter XIX. is occupied with the easier forms of Differential Equations. The last chapter contains a distinct innovation in the form of the more elementary principles of Graphics and Nomography.

The book is not a mere engineering manual, but is intended mainly for students who desire to obtain a sound academic knowledge of the subject. It gives plenty of practical applications on which the student can exercise himself, and it does not "hedge" the fundamental principles of Number, Continuity, etc., on which the modern conception of the Calculus is based; but wherever these abstract motions are introduced, Prof. Carey attempts by concrete illustration to vivify and vitalise as far as possible those nebulous entities which in many modern treatises and lectures leave the ordinary student vague and miserable. Graphical illustrations are freely used throughout the book. We would especially commend to the reader's notice page 261, in which the various polynomials x, $x-\frac{x^3}{6}$, $x-\frac{x^3}{6}+\frac{x^5}{120}$, etc., are actually sketched and exhibited as functions approximating more and more closely to $\sin x$. Thus the beginner can see with great vividness for what range of values of x the various polynomials can be regarded as serious approximations to the function $\sin x$ itself. Thus the basic ideas of Convergence and Range of Validity present themselves in shadowy forms.

It is impossible at this stage in the development of the teaching of the Calculus to appraise with accuracy and justice the true value and practicability of Prof. Carey's presentation of the subject. What is certain is, that he has written a most valuable and noteworthy book, and one which will well repay reading. On the other hand, modern pedagogy is almost unanimous in thinking that the best plan for any student is to proceed in his first reading fast and far into the depths of the subject which he is studying, using only intuition as far as possible and taking much for granted. After thus getting a bird's-eye view of the mathematical territory to be explored, he is in a position to re-traverse the ground more carefully, studying the philosophical bases on which the subject rests and trying to discover the minimum number of axioms. Chrystal states this pedagogical principle very clearly in the Preface to his *Algebra*, and we have recently had another explicit statement on the same subject by Sir Ronald Ross in his presidential lecture to the Science Masters' Association. It will probably be found best in the long-run, therefore, to make one's first perusal of the Calculus graphical and intuitional, paying great heed to mechanical processes and practical applications such as Mechanics and Geometry. The student is then in a position to pursue a later course of study, in which he pays little heed to processes and applications, but devotes all his attention to concepts of "Number" and "Continuity," rigorous theorems on "Limits," and so forth,—in fact, the modern courses on the so-called subject of "Epsilonology" as set forth in treatises on "Analysis," "Sets of Points," etc. Prof. Carey's book is not based on this view; he aims at a high standard of rigour from the outset, but *quot homines, tot sententiae*, and the two volumes will well repay reading by all whom it may concern.

WILLIAM P. MILNE.

The Organisation of Thought, Educational and Scientific. By A. N. WHITEHEAD. Pp. 228. 6s. net. 1917. (Williams and Norgate.)

The volume before us, which has been recently placed on the shelves of the Library of the Association, seems to us in many respects one of the most helpful and stimulating contributions towards our own problems of reconstruction that we have read. It consists of eight addresses, of which "five deal with education, and the remaining three embody discussions on certain points arising in the philosophy of science." Such a collection is often open to the reproach of being an unwarrantable piece of book-making—a reproach anticipated by the author. "A common line of reflection extends through the whole. . . . The various parts of the book were . . . composed with

express reference to each other, so as to form a whole." One of the discourses is here published for the first time. It is that on "The Anatomy of Some Scientific Ideas."

When we consider the natural temperament of the man who is happily permitted by aptitude and circumstance to initiate and develop a line of research demanding the closest application and a degree of intellectual effort that must at times be nothing less than exhausting, it is clear that nothing but the strongest of motives will prove the "spear to prick the sides of his intent," and to induce him to forsake for the time his immediate and absorbing interests. Such a man was the lamented Fitzgerald, lost to us all too soon, of whom it was said that possibly he had "a real and semi-inspired insight into the inmost processes of Nature." Sir Oliver Lodge has told us that this great physicist had "with complete sincerity" made up his mind, under the potent stimulus of urgent public need, to abandon research, and to devote himself to the organisation of National Education. He felt that "whether the human race got to know about the ether now or fifty years hence was a small matter, but whether the present state of appalling scientific ignorance was to continue for another generation was a vital matter affecting the future of his country in a positive and definite way."

Our President, Prof. Whitehead, came to us from those lofty peaks of intellectual endeavour, which have been scaled by few, to which but few aspirants can hope to climb, and which the mass of us must be content to contemplate from the far-off plains below. The address to us in 1916, with which this volume opens, revealed to us a leader whose vision was as penetrating as his outlook was comprehensive. The apathy of the general public to educational questions had been roughly shaken by events. He took a wide and luminous view of the tempest of controversy which was then raging. In the presence of a great national emergency the apparently unending dispute assumed to him the dimensions of a squabble between Big-Endians and Little-Endians. The acres of annual invective might as well be utilised, as our ancestors would say, for "pye-bottoms." With unerring instinct he swept away the non-essentials and took us down to the bed-rock of things. This quarrel over a little more or less Latin or Greek betokens, said he, a provincial attitude towards the problem before us. What we see is the accomplished fact. The truth is that "we are witnessing the dying away of the classical impulse which has dominated European thought since the time of the Renaissance." Hence it is idle to listen to "great Argument about it and about."

Three fundamental changes have made between the life of the past and that of to-day a gap that cannot be bridged. "Science now enters into the very texture of our thoughts; its methods and results colour the imaginations of our poets; they modify the conclusions of philosophers and theologians. Again, mechanical inventions, which are the product of science, by altering the material possibilities of life have transformed our industrial system, and thus changed the very structure of Society. Finally, the idea of the world now means to us the whole round world of human affairs. . . . The total result of these changes is that the supreme merit of immediate relevance to the full compass of modern life has been lost to classical literature. Whether we regret it or no, the absolute dominance of classical ideas in education is necessarily doomed." Let us now set our own house in order: it needs it.

He condenses all the Law and the Prophets into two commandments. "Do not teach too many subjects." That we shall obey as soon as the devil of scraps and snippets is exorcised from the scholastic world. "What you teach, teach thoroughly." To obey that in its fullest sense we cannot until we are relieved of an age-long incubus, and schools, not scholars, are examined. Then at last, as free men and women, we can embark with some confidence on the great crusade to which he calls us, against the greatest of all educational pests—the INERT IDEA. "Every intellectual revolution which has ever stirred humanity into greatness has been a passionate protest against inert ideas." Here we must beware lest in our eagerness to get rid of what is inherently barren, or actively sterilising, we may in our enthusiasm, in our ignorance and inexperience, replace what is so harmful by what may in its turn prove to be every whit as lacking in vitality. The rush for Graphs will

serve as an instance. Unhappily, this tended to the adoption of Graphs, "with no sort of idea behind them, but just graphs." Educational schedules must not be reformed "without a clear conception of the attributes which you wish to evoke in the living minds of the children. . . . You cannot put life into any schedule of general education unless you succeed in exhibiting its relation to some essential characteristic of all intelligent or emotional perception. . . . Reformation must begin at the other end." Here we are brought to a distinctive characteristic of Prof. Whitehead's teaching—the stress he lays on such quantitative aspects of the world as are simple enough to be brought within the purview of the young. "Elegant intellects which despise the theory of quantity are but half-developed," and "more to be pitied than blamed."

Quantity and number face us everywhere. Here then is your opportunity. You may, for instance, frame your schedule of algebra so as to exhibit the lessons of history. Do not disinter dry bones. "The curves of history are more vivid and informing than lists of names and dates. What purpose is effected by a catalogue of undistinguished kings and queens? Tom, Dick, or Harry, they are all dead. . . . General resurrections are failures, and are better postponed." So go to what is alive. "The quantitative flux of the forces of modern society is capable of very simple exhibition." And *pari passu* as an abstract science other ideas are being studied for their own sake—the variable, the function, rate of change, equations and their solution, and elimination. We had almost written for this last word—extermination, by which forcible name the process was known, at any rate across the Tweed, in the early years of the last century. In the typical application of the algebra schedule selected by the author two points occur. A simple exhibition of the quantitative flux of the forces of modern society presupposes a liberality of outlook in the authorities of public schools which we fear does not exist—at any rate in few. "The route from Chaucer to the Black Death, from the Black Death to modern Labour Troubles" will be the last straw—what is now-a-days elegantly called "THE limit"—in schools which would regard with hostility the teaching of elementary politics even by their own carefully selected staff. And in the second place, what proportion of our mathematical masters is at present competent by training and width of interests to undertake such tasks with success? Prof. Whitehead meets the objection in the only way. The process of exhibiting the "applications of knowledge must . . . essentially depend on the . . . judgment and genius of the teacher." We cannot get over the fact that "pupils have got to be made to feel that they are studying something, and are not merely executing intellectual minuets." Poincaré somewhere marvels that there are types of mind to which mathematics is entirely repugnant. Prof. Whitehead admits that the subject is the very typical example of reconditeness—"not of difficulty, but the ideas involved are of highly special applications, and rarely influence thought." It has been said that, considering the barren formalism and the teaching of the subject, it is a miracle not merely that so few mathematicians are produced but that the born mathematician survives. With the latter part of the statement we are not here concerned. But the former concerns us more than anything else. The fact is that "the subject, as it exists in the minds and in the books of students of mathematics, *is* recondite." For the purposes of general education it must "be subject to a rigorous process of selection and adaptation," and in these addresses we find indicated the broad lines such a process may follow. Prof. Whitehead pays a well-deserved compliment to the present generation of teachers for the amount they have accomplished since the work of reform was initiated. The essential thing is that the ideal shall take its shape in the minds of the mass of teachers. "All recent experience has shown that the majority of teachers are only too ready to welcome any practicable means of rescuing the subject from the reproach of being a mechanical discipline."

If these few remarks have been largely based upon the two addresses which many of us have heard, it is because a very large number of our members were at the time of their delivery in other lands, while others had preoccupations to which we need do no more than allude. Suffice it to say that the remaining addresses are also compact with wit and wisdom, and that there are few pages

within the covers of the book that do not contain something to arrest, some sentence or phrase which will serve as a text for "furious thinking." Nor are there lacking noble thoughts, finely expressed in "words that burn." We hope that it is one of those books which will not be found upon the teacher's shelves until he has read, marked, digested, and enjoyed it from end to end. It is not often that we are privileged to accompany a thinker whose earnestness is so notably tempered by one of the greatest of virtues—σωφροσύνη.

Œuvres de Charles Hermite, publiés sous les auspices de l'Académie des Sciences. Par Emile Picard. Tom. 4. Pp. vi+593. 25 frs. 1917. (Gauthier-Villars.)

With this volume of some ninety papers, published between 1880 and 1901, come to a close the pious labours of M. Picard in erecting a fit monument to Hermite. The four volumes, taken with the Letters of Hermite and Stieltjes published some years ago, and Prof. Mansion's biographical sketch and bibliography, will form an almost complete record of the long and busy life of the famous mathematician. Portraits have been reproduced for all the previous volumes, and in this we have a reproduction of the beautiful medal due to J. C. Chaplain, struck on the occasion of Hermite's seventieth anniversary. If we remember that the claims of his official duties during the period covered by these papers were of the most exacting character, it is not surprising to find that by far the majority of them are short. A very considerable number of them deal with elliptic functions, the subject on which he had laid down the lines for his future researches at the early age of twenty-eight. In such papers and memoirs as these, he found relaxation amid the manifold preoccupations in which his later years were largely absorbed. It may not be untimely to remind the reader of the especial importance attached by Hermite to abstraction as an invaluable quality in the equipment of those who may be called upon to serve their country in the field. He regarded rigorous mental discipline as the essential preparation for military duties, and maintained that mathematics, more than any other subject of study, cultivated the power of abstraction which is so indispensable to the leader in framing his course of action amid the obscurity and tumult of a battle. And he saw the influence of such studies extending even further in its power to mould the deepest and most sacred conceptions of the human mind. For instance, in a speech on the occasion of his Jubilee, he classed together the two great schools—the École Normale and the École Polytechnique—as two branches of the same family, inseparably united by the common sense of justice and duty—a sense which, in some way which we cannot fathom, is intimately connected with the science of mathematics, and seems to pass from the intellect to the consciousness, upon which it is imposed with all the force of the truths of Geometry. We find scattered throughout this volume the various obituary notices and addresses which he was so often called upon to prepare from his official position and as the *doyen* of French mathematicians, and from these it would be easy to fill a number of the *Gazette* with purple patches. It speaks volumes for the man that there are few if any of these from which the human touch is missing, connected as he was with so many of the mathematicians of his day by ties of long friendship. With masterly skill he traces in turn the main features of the life-work of each, its scope, and its place among contemporary contributions to mathematical science. Then he says, of a Halphen: "But as we stand by his grave and speak of his work we remember the colleague, the friend we have lost. His simplicity and modesty were on a par with his genius; he was kind and affectionate, he was untiring in his devotion to his duty. While quite a young officer he was attached to the Army of the North, was made Captain and decorated on the field of battle, at Pont-Noyelles. He was present at the battle of Bapaume. This great mathematician was a soldier. To him we tender the supreme homage of our admiration for his work, our sorrow for the loss we have sustained, and our expression of the affectionate memories which we shall cherish to our latest breath." Or again, of a Kronecker: "But the tongue of praise is stilled in the presence of the sorrows of Science and the emotion caused throughout the world of Mathematics by the cruel loss of

this great mathematician. To that sorrowful regret, to those recollections of a laborious life crowned by discoveries of such value, I now add those of a friendship which for thirty years has been the honour of my scientific career, a friendship the like of which it will never again be mine to enjoy." Or of our own Cayley: "I had some share in several of the investigations undertaken by Cayley: the same problems brought us together at the beginning of our career, and I shall never forget his kindness, his great simplicity of character, his absolute devotion to our Science. To that remembrance, which is very dear to me, I add my sorrowful regret, and the homage that I offer to his memory." And of Weierstrass, he said: "The life of our illustrious Colleague has been entirely consecrated to the Science which he served with an absolute devotion. It has been a long life, and full of honours; but as we stand at the brink of the grave which is about to close over him, the sole thought that fills our minds is that of his genius and that universal respect which is due from all men to nobility of character. Weierstrass was a good and upright man; be his guerdon our last tribute of respect to his memory! That memory will live as long as there are men who are eager in the quest for Truth, as long as men can be found to devote their labours to fresh efforts for the advancement of Analysis, and for the progress of the Science of Calculation." Finally, let us quote from his short tribute to his friend Brioschi: "I have shared in the labours of Brioschi: and often our investigations have been directed to a common end. I have followed his noble career, so notable for his unceasing toil and for the great services he has rendered to his country. None can feel more than I the loss of that great mathematician, of that man of stainless honour. The memory of our friendship, and of an intimacy which dates back to our early youth, will ever remain to me as one of the dearest amd most cherished possessions of my life." The reader must not fail to read the two addresses delivered respectively at the opening of the new Sorbonne, and at the Académie des Sciences in 1889 and 1890, containing as they do a number of exquisitely conceived pen-pictures of the mathematical and other worthies whom it became the occasion to honour. The volume closes with the affecting speech made by Hermite as the recipient of the Chapelin medal on the occasion of his Jubilee in 1892, and with facsimiles of two pages from a letter to Jules Tannery.

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